



# THE STABILITY OF CYLINDRICAL SHELLS JOINED AT AN ANGLE UNDER THE ACTION OF A UNIFORM EXTERNAL PRESSURE†

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An asymptotic method [1] is used for the approximate solution of the problem of the loss of stability of the zero-moment stressed state of two thin closed elastic cylindrical shells of average length under the action of a uniform external lateral pressure. Simple approximate formulae are obtained for determining the upper critical pressure and the form of the loss of stability. It is established that the form of the loss of stability is localized in the neighbourhood of the longest generatrix of the shell which has the greater length, and that the upper critical pressure differs only slightly from the critical pressure for the longer shell supported by a hinge along the joining line.

1. Let us assume that the cylindrical shells, joined at an angle  $2\beta$ , have the same radius  $R$  and thickness  $h$ . We denote by  $x^{(k)}, \varphi^{(k)}$  the system of coordinates in the median surface of the  $k$ th shell. Suppose

$$\begin{aligned} \varphi^{(1)} = \varphi^{(2)} = \varphi, \quad x^{(1)} \in [x_0^{(1)}, x_*^{(1)}(\varphi)], \quad x^{(2)} \in [x_*^{(2)}(\varphi), x_0^{(2)}] \\ x_*^{(k)}(\varphi) = x_0^{(k)} + (-1)^{k+1}(l_c^{(k)} + \text{tg}\beta \cos \varphi), \quad x_0^{(1)} + l_c^{(1)} = x_0^{(2)} - l_c^{(2)} \\ l_c^{(k)} = L_c^{(k)} / R, \quad k = 1, 2 \end{aligned}$$

where  $L_c^{(k)}$  is the “average” length of the  $K$ th shell (Fig. 1).

Here and henceforth, quantities with a superscript  $k$  correspond to the  $k$ th shell. Superscripts may be omitted from formulae which hold for both shells.

The equations of stability for each of the shells can be written in the following dimensionless form [2]

$$\begin{aligned} T_2 - Q_{2,\varphi} - Q_{1,x} + \lambda(w_{,\varphi\varphi} - v_{,\varphi}) = 0, \quad S_{,x} + T_{2,\varphi} + Q_2 = 0 \\ T_{1,x} + S_{,\varphi} = 0, \quad Q_2 = M_{2,\varphi}, \quad Q_1 = M_{1,x} + 2H_{,\varphi}, \quad H = (1 - \nu)\varepsilon^8 \vartheta_{2,x} \\ M_1 = \varepsilon^8 (\vartheta_{1,x} + \nu \vartheta_{2,\varphi}), \quad M_2 = \varepsilon^8 (\vartheta_{2,\varphi} + \nu \vartheta_{1,x}) \tag{1.1} \\ (1 - \nu^2)T_1 = u_{,x} + \nu(w + v_{,\varphi}), \quad (1 - \nu^2)T_2 = w + v_{,\varphi} + \nu u_{,x} \\ 2(1 - \nu)S = v_{,x} + u_{,\varphi}, \quad v_1 = -w_{,x}, \quad v_2 = -w_{,\varphi} + v \end{aligned}$$

Here,  $T_1, T_2, S, Q_1, Q_2, M_1, M_2, H$  are the dimensionless forces and moments,  $\vartheta_1$  and  $\vartheta_2$  are the angles of rotation,  $u, v$  and  $w$  are the dimensionless projections of the variables,  $\varepsilon^8 = h_2/[12(1 - \nu^2)R^2]$  is a small parameter,  $p$  is the value of the external pressure,  $\nu$  is Poisson’s ratio and  $E$  is Young’s modulus.

Let us assume that the straight edges of the shells are freely supported

$$T_1^{(k)} = v^{(k)} = w^{(k)} = M_1^{(k)} = 0, \quad x^{(k)} = x_0^{(k)} \tag{1.2}$$

Along the line where the shells are joined  $x^{(1)} = x_*^{(1)}, x^{(2)} = x_*^{(2)}$ , the following continuity conditions must be satisfied:

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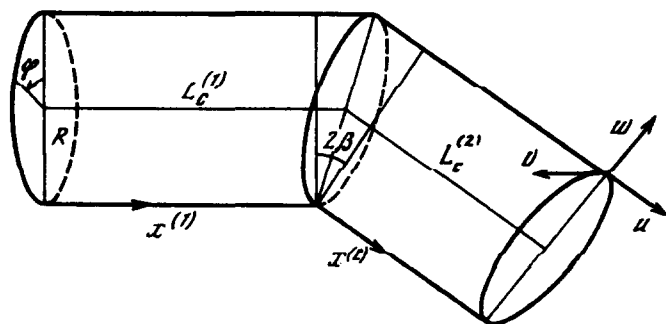


Fig. 1.

(a) for the displacements

$$\begin{aligned} u_n^{(1)} c \sin \beta + w^{(1)} \cos \beta &= -u_n^{(2)} c \sin \beta + w^{(2)} \cos \beta \\ u_n^{(1)} \cos \beta - w^{(1)} c \sin \beta &= u_n^{(2)} \cos \beta + w^{(2)} c \sin \beta \\ u_i^{(1)} &= u_i^{(2)} \end{aligned} \quad (1.3)$$

where

$$u_n^{(k)} = u^{(k)} \cos \gamma + (-1)^{k+1} v^{(k)} \sin \gamma, \quad u_i^{(k)} = (-1)^k u^{(k)} \sin \gamma + v^{(k)} \cos \gamma$$

$$\operatorname{tg} \gamma + \operatorname{tg} \beta \sin \varphi, \quad c = \cos \gamma \cos \varphi$$

(b) for the forces

$$\begin{aligned} T_n^{(1)} c \sin \beta + Q_n^{(1)} \cos \beta &= -T_n^{(2)} c \sin \beta + Q_n^{(2)} \cos \beta \\ T_n^{(1)} \cos \beta - Q_n^{(1)} c \sin \beta &= T_n^{(2)} \cos \beta + Q_n^{(2)} c \sin \beta \\ S_{nr}^{(1)} &= S_{nr}^{(2)} \end{aligned} \quad (1.4)$$

where

$$\begin{aligned} T_n^{(k)} &= T_1^{(k)} \cos^2 \gamma + 2(-1)^{k+1} S^{(k)} \sin \gamma \cos \gamma + T_2^{(k)} \sin^2 \gamma \\ S_{nr}^{(k)} &= (-1)^{k+1} (T_2^{(k)} - T_1^{(k)}) \sin \gamma \cos \gamma + S^{(k)} (\cos^2 \gamma - \sin^2 \gamma) \\ Q_n^{(k)} &= Q_1^{(k)} \cos \gamma + (-1)^{k+1} Q_2^{(k)} \sin \gamma \end{aligned}$$

(c) for the angles of rotation and moments

$$\vartheta_n^{(1)} = \vartheta_n^{(2)}, \quad M_n^{(1)} = M_n^{(2)} \quad (1.5)$$

where

$$\begin{aligned} M_n^{(k)} &= M_1^{(k)} \cos^2 \gamma + 2(-1)^{k+1} H^{(k)} \sin \gamma \cos \gamma + M_2^{(k)} \sin^2 \gamma \\ \vartheta_n^{(k)} &= \vartheta_1^{(k)} \cos \gamma + (-1)^{k+1} \vartheta_2^{(k)} \sin \gamma \end{aligned}$$

Certain small terms, which have no effect on the approximate formulae obtained in this paper have been discarded in relationships (1.1)–(1.5). For instance, the effect of subcritical deformations turns out to be unimportant. However, it did turn out that terms have to be retained in (1.1) which do not appear in the technical theory of Donnel and Vlasov for determining the critical pressure with the accuracy adopted in this paper.

It is henceforth assumed that the joining angle  $\beta$  is not a small quantity, that is,  $\beta = O(1)$ .

2. We shall seek an approximate solution of the boundary-value problem (1.1)–(1.5), which falls off exponentially on moving away from the generatrix  $\varphi = 0$ , in the form

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots \tag{2.1}$$

$$y = y_a + y_b \tag{2.2}$$

Here,  $y$  is any of the unknown functions occurring in (1.1). The first term in (2.2) describes the basic semi-zero-moment stress-strain state of the shell and the second describes the simple edge effect close to the edges  $x^{(k)} = x_0^{(k)}$  and the joining line  $x^{(k)} = x_*^{(k)}$ . Let us represent the approximate solution  $y_a$  in the form [1]

$$y_a = \varepsilon^{a(y)} \tilde{y} \exp\left(\frac{i}{\varepsilon} \int_0^\varphi q(\varphi) d\varphi\right) \tag{2.3}$$

where

$$\text{Im}\{q(0)\} = 0, \quad \text{Im}\{q_{,\varphi}(0)\} > 0 \tag{2.4}$$

$$\tilde{y} = \sum_{n=0}^{\infty} \varepsilon^n y_n(x, \varphi) \tag{2.5}$$

In order to determine the intensity factor  $a(y)$ , the constants  $\lambda_n$  and the functions  $q(\varphi)$ ,  $y_n(x, \varphi)$ , we substitute (2.1) and (2.3) into (1.1). Let us put  $a(w) = 0$ . Then,  $a(v) = 1$ ,  $a(u) = a(T_1) = 2$ ,  $a(S) = 3$  and the function  $\tilde{w}$  satisfies the equation

$$G\tilde{w} - \lambda\tilde{w} = \varepsilon^2 N\tilde{w}, \quad N = 4 \frac{\partial^2}{\partial x^2} + 2 - \lambda_0 \Phi^{-2} \left(1 + 2 \frac{\partial^2}{\partial x^2}\right) \tag{2.6}$$

$$G = \Phi^{-6} \frac{\partial^4}{\partial x^4} + \Phi^2, \quad \Phi = q - i\varepsilon \frac{\partial}{\partial \varphi}$$

Here,  $\Phi^{-n}$  is a formal asymptotic series and  $\Phi^{-n}\Phi^n y \approx y$ .

The quantities  $\tilde{u}$ ,  $\tilde{v}$ , ... are expressed in terms of  $\tilde{w}$ . In particular

$$\begin{aligned} \tilde{u} &\approx -\Phi^{-2} \tilde{w}_{,x} - \varepsilon^2 (2 + \nu) \Phi^{-4} \tilde{w}_{,xxx}, & \tilde{T}_1 &\approx -\Phi^{-2} \tilde{w}_{,xx} + 2\varepsilon^2 (q^4 - \lambda_0 q^2) \tilde{w} \\ \tilde{v} &\approx i\Phi^{-1} \tilde{w} - i\nu\varepsilon^2 \Phi^{-3} \tilde{w}_{,xx}, & \tilde{S} &\approx -i\Phi^{-3} \tilde{w}_{,xxx} + 2i\varepsilon^2 q (q^2 - \lambda_0) \Phi^{-1} \tilde{w}_{,x} \end{aligned}$$

We expand the operator  $G$ , defined in (2.6) in series in powers of the parameter  $\varepsilon$

$$\begin{aligned} G &= \sum_{n=0}^{\infty} \varepsilon^n G_n, \quad G_0 = \frac{1}{q^6} \frac{\partial^4}{\partial x^4} + q^2, \quad G_1 = -i \left( \frac{\partial G_0}{\partial q} \frac{\partial}{\partial \varphi} + \frac{q_{,\varphi}^2}{2} \frac{\partial^2 G_0}{\partial q^2} \right) \\ G_2 &= -\frac{1}{2} \left( \frac{\partial^2 G_0}{\partial q^2} \frac{\partial^2}{\partial \varphi^2} + q_{,\varphi} \frac{\partial^3 G_0}{\partial q^3} \frac{\partial}{\partial \varphi} + \frac{q_{,\varphi\varphi}}{3} \frac{\partial^3 G_0}{\partial q^3} + \frac{(q_{,\varphi})^2}{4} \frac{\partial^4 G_0}{\partial q^4} \right) \end{aligned} \tag{2.7}$$

Substituting (2.1), (2.5) and (2.7) into (2.6), we obtain a sequence of equations for determining  $\lambda_n$  and  $w_n(x, \varphi)$

$$(G_0 - \lambda_0)w_0 = 0, \quad (G_0 - \lambda_0)w_1 + (G_1 - \lambda_1)w_0 = 0, \dots \tag{2.8}$$

The edge effect functions  $y_b$  can be represented in the form

$$\begin{aligned} y_b^{(k)} &= \varepsilon^{b(y)} \hat{y}^{(k)} \exp\left(\frac{i}{\varepsilon} q_0 \varphi\right), \quad \hat{y}^{(k)} = \sum_{j=1}^4 C_j^{(k)} \hat{y}_j f_j^{(k)}(x, \varepsilon) \\ f_j^{(k)} &= \exp[(-1)^k (g_j^{(k)} - x^{(k)}) r_j / \varepsilon^2], \quad j = 1, 2, 3, 4 \\ g_1^{(k)} &= g_2^{(k)} = x_0^{(k)}, \quad g_3^{(k)} = g_4^{(k)} = x_*^{(k)}(0) \\ r_j &= -[1 + (-1)^j i] / \sqrt{2}, \quad j = 1, 2, \quad r_3 = -r_2, \quad r_4 = -r_1 \end{aligned} \tag{2.9}$$

where  $\hat{y}_j, C_j^{(k)}$  are unknown constants and  $q_0 = q(0)$ . The functions  $f_1^{(k)}, f_2^{(k)}$  become rapidly smaller as the distance from the straight edge of the  $k$ th shell becomes larger, and the functions rapidly decrease as the distance from the joining line increases.

By substituting (2.9) into system (1.1) we can find the relationship between the intensity factors  $b(y)$  and determine  $\hat{y}_j$ . In particular

$$\begin{aligned} \hat{w}_j &= 1, \quad \hat{\vartheta}_{nj} = -r_j, \quad \hat{M}_{nj} = -r_j^2 \\ \hat{Q}_{nj} &= -r_j^3, \quad \hat{S}_{nj} = \frac{iq_0}{r_j}, \quad \hat{T}_{nj} = -\frac{q_0}{r_j^2} + \frac{\text{tg}\beta}{r_j} \end{aligned}$$

The actual values of the intensity factors

$$\begin{aligned} b(w) &= 2, \quad b(\vartheta_n) = 0, \quad b(M_n) = 6 \\ b(Q_n) &= 4, \quad b(S_n) = 3, \quad b(T_n) = 4 \end{aligned}$$

are found after substituting the solution (2.2) into the boundary conditions (1.2)–(1.5). The same substitution enables us to determine the boundary conditions for Eq. (2.8). In the zeroth and first approximations these conditions are separated into two groups, each of which contains only quantities which refer to one of the shells. This enables us to solve the boundary-value problems of the zeroth and first approximations separately for each of the shells.

3. We will now solve the boundary-value problems for Eq. (2.8) in succession. In the zeroth approximation, we obtain two independent boundary-value problems

$$\begin{aligned} (G_0 - \lambda_0)w_0^{(k)} &= 0 \\ w_0^{(k)} = w_{0,xx}^{(k)} &= 0 \quad \text{when } x^{(k)} = x_0^{(k)}, \quad x^{(k)} = x_*^{(k)} \end{aligned} \quad (3.1)$$

which are identical with the boundary-value problems of the zeroth approximation for the first shell ( $k = 1$ ) and the second shell ( $k = 2$ ) with freely supported edges.

The solution of each of problems (3.1) will be sought in the form

$$w_0(x, \varphi) = P_0(\varphi)W_0(x, \varphi)$$

Here

$$P_0(\varphi) = 1 + V_1\varphi + V_2\varphi^2 / 2 + \dots \quad (3.2)$$

and the coefficients  $V_n$  are determined when constructing the first approximation. The function  $W_0$  satisfied the equation

$$d^4 W_0 / dx^4 - \alpha^4 W_0 = 0, \quad \alpha^4 = \lambda_0 q^6 - q^8 \quad (3.3)$$

and the boundary conditions

$$W_0 = W_{0,xx} = 0 \quad \text{when } x = x_0, \quad x = x_* \quad (3.4)$$

It follows from (3.3) and (3.4) that

$$\lambda_0 = \alpha^4 / q^6 + q^2 = f(\varphi, q(\varphi)) \quad (3.5)$$

where  $\alpha = \pi/l(\varphi)$  is the least positive eigenvalue of boundary-value problem (3.3), (3.4),  $l(\varphi) = l_c = \text{tg} \beta \cos \varphi$ .

The values  $\varphi = 0$  and  $q = q_0 = q(0)$  for which

$$\lambda_0 = f(0, q_0) = \min_q f(0, q) \quad (3.6)$$

correspond to the zeroth approximation for the critical loading parameter  $\lambda_0$ .

It follows from (3.6) and (2.4) that

$$q_0 = 3^{1/8} \sqrt{\alpha_0}, \quad \alpha_0 = \pi / l_0, \quad l_0 = l(0) = l_c + \text{tg}\beta \quad (3.7)$$

Substituting (3.7) into (3.6), we obtain

$$\lambda_0 = 4\pi / 3^{3/4} l_0 \quad (3.8)$$

We will represent  $q(\varphi)$  in the form of the series

$$q = \sum_{n=0}^{\infty} \frac{q_n \Phi^n}{n!}$$

where  $q_0$  is defined by (3.7). In order to find the remaining coefficients  $q_n$ , it is necessary to differentiate (3.5) with respect to  $\varphi$  and then put  $\varphi = 0$ . In particular

$$q_1 = \frac{i}{4} \sqrt{\frac{\lambda_0 \text{tg}\beta}{l_0}}, \quad q_2 = -\frac{11q_1^2}{3q_0}, \quad q_3 = -\frac{q_1}{3} \left( \frac{40q_1^2}{q_0^2} + \frac{3}{4} \right) \quad (3.9)$$

Without loss of generality, it may be assumed that  $l_0^{(1)} \geq l_0^{(2)}$ . Then  $l_0^{(1)} \geq l_0^{(2)}$  and, consequently, the critical pressure parameter for the joined shells is calculated in the zeroth approximation using the formula

$$\lambda \approx \lambda_0^{(1)} = 4\pi / (3^{3/4} l_0^{(1)}) \quad (3.10)$$

The eigenfunctions of boundary-value problem (3.3), (3.4), corresponding to  $\lambda_0^{(1)}$ , can be represented in the form

$$W_0^{(1)} = D^{(1)} \sin[\alpha^{(1)}(x^{(1)} - x_0^1)], \quad W_0^{(2)} = D^{(2)} \sin[\alpha^{(2)}(x_0^2 - x^{(2)})]$$

where  $D^{(k)}$  are arbitrary constants.

In the non-resonant case

$$l_0^{(1)} - l_0^{(2)} \sim 1 \quad (3.11)$$

the boundary-value problem (3.3), (3.4), when  $k = 2$ , has only a trivial solution and hence  $D^{(2)} = 0$ .

In the resonant case

$$l_0^{(1)} = l_0^{(2)} + \varepsilon^2 \Delta, \quad \Delta = O(1) \quad (3.12)$$

and the ratio  $D^{(2)}/D^{(1)}$  is determined when constructing the following approximations.

4. The boundary-value problem of the first approximation for each of the shells has the form

$$(G_0 - \lambda_0)w_1 = -(G_1 - \lambda_1)w_0 \quad (4.1)$$

$$w_1 = w_{1,xx} = 0 \text{ when } x = x_0 \quad (4.2)$$

$$w_1 = w_{1,xx} - 4iq^{-1}w_{0,xxx}x^*,\varphi = 0 \text{ when } x = x_*$$

and is an inhomogeneous spectral problem. We obtain the condition for it to be solvable by multiplying Eq. (4.1) by  $W_0$  and integrating the resulting equality by parts over the range of variation of  $x$ , taking account of (4.2). After some reduction, the condition of solvability takes the form

$$f_{,q}(P_{0,\varphi}I_0 + P_0I_1) + (f_{,q\varphi} + f_{,qq}q_{, \varphi} - 2\lambda_1i)I_0P_0 / 2 = 0 \quad (4.3)$$

$$I_0 = \int_{x_0}^{x_*} W_0^2 dx, \quad I_1 = \int_{x_0}^{x_*} W_0 W_{0,\varphi} dx$$

In view of the fact that  $f_{,q}(0, q_0) = f_{,q\varphi}(0, q_0) = 0$ , it follows from (4.3) that

$$\lambda_1 = -iq_1 f_{,qq}(0, q_0) / 2 = 2\sqrt{\lambda_0 \operatorname{tg} \beta} / l_0 \quad (4.4)$$

The coefficients  $V_n$  in the expansion (3.2) of the function  $P_0(\varphi)$  are successively determined by calculating the derivatives with respect to  $\varphi$  of the left-hand side of (4.3) when  $\varphi = 0$ . Taking account of (3.9), we obtain

$$V_1 = \frac{4q_1}{3q_0}, \quad V_2 = \frac{103q_1^2}{18q_0^2} + \frac{1}{16} \quad (4.5)$$

It follows from (4.1) and (4.3) that

$$(G_0 - \lambda_0)w_1 = -\frac{2i}{q} f_{,\varphi} P_0 W_0 + if_{,q} P_0 \left( W_{0,\varphi} + \frac{\alpha}{2\alpha} W_0 \right) \quad (4.6)$$

The solution of (4.6) will be sought in the form

$$w_1 = P_1(\varphi)W_0 + A(\varphi)W_{0,\varphi\varphi} + B(\varphi)W_{0,\varphi} + C(\varphi)\sinh[\alpha(x - x_0)] \quad (4.7)$$

We substitute (4.7) into (4.6) and use the equalities

$$(G_0 - \lambda_0)W_{0,\varphi} = f_{,\varphi} W_0, \quad (G_0 - \lambda_0)W_{0,\varphi\varphi} = 2f_{,\varphi} W_{0,\varphi} + f_{,\varphi\varphi} W_0$$

For the functions  $A$  and  $B$ , we obtain the expressions

$$A = \frac{if_{,q}}{2f_{,\varphi}} P_0, \quad B = -A \left( 4 \frac{x_{*,\varphi}}{l} + \operatorname{tg} \varphi \right) \quad (4.8)$$

The function  $C$  is determined after substituting (4.7) into (4.2) and taking account of (4.8), and the functions  $P_1(\varphi)$  are found when constructing the next approximation.

For the parameter  $\lambda$ , when account is taken of the correction of the first approximation, we obtain

$$\lambda \approx \lambda_0^{(1)} + 2\varepsilon \sqrt{\lambda_0^{(1)} \operatorname{tg} \beta} / l_0^{(1)} \quad (4.9)$$

Formula (4.9) is identical with that obtained in [1] for a shell with freely supported edges. Consequently, the parameter  $\lambda$  for joined shells differs from the value of  $\lambda$  for the longer shell with freely supported edges by an amount of the order of  $O(\varepsilon^2)$ .

5. To determine the quantity  $\lambda_2$ , it is sufficient to consider the boundary-value problem of the second approximation when  $\varphi = 0$

$$\begin{aligned} (G_0 - \lambda_0)w_2^{(k)} + (G_1 - \lambda_1)w_1^{(k)} + (G_2 - \lambda_2)w_0^{(k)} &= Nw_0^{(k)} \\ w_2^{(k)} &= -\hat{w}^{(k)}, \quad w_{2,xx}^{(k)} = s_0 \hat{T}_n^{(k)} \quad \text{when } x^{(k)} = x_0^{(k)} \\ w_2^{(k)} &= \frac{w_{0,x}^{(2)} - w_{0,x}^{(1)} \cos 2\beta}{s_0 \sin 2\beta} - \hat{w}^{(1)}, \quad w_2^{(2)} = \frac{w_{0,x}^{(2)} \cos 2\beta - w_{0,x}^{(1)}}{s_0 \sin 2\beta} - \hat{w}^{(2)} \\ w_{2,xx}^{(1)} &= s_0 \left( \hat{T}_n^{(1)} + \frac{\hat{Q}_n^{(1)} \cos 2\beta - \hat{Q}_n^{(2)}}{\sin 2\beta} \right) - \frac{5}{s_0} w_{0,xx}^{(1)} \operatorname{tg} \beta \\ w_{2,xx}^{(1)} &= s_0 \left( \hat{T}_n^{(2)} - \frac{\hat{Q}_n^{(2)} \cos 2\beta - \hat{Q}_n^{(1)}}{\sin 2\beta} \right) + \frac{5}{s_0} w_{0,xx}^{(2)} \operatorname{tg} \beta \end{aligned} \quad (5.1)$$

$$\text{when } x^{(k)} = x_*^{(k)}(0), \quad s_0 = (q_0^{(1)})^2$$

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The boundary conditions contain the values of the edge-effect functions.

Substituting solution (2.2) into (1.1)–(1.5) we obtain the following relationships for determining the constants  $C_j^{(k)}$  in the edge-effect functions (2.9)

$$\hat{w}^{(k)} = \hat{M}_n^{(k)} 0 \text{ when } x^{(k)} = x_0^{(k)} \tag{5.2}$$

$$\hat{w}^{(1)} = \hat{w}^{(2)}, \quad \hat{\vartheta}^{(1)} - \hat{\vartheta}^{(2)} = w_{0,x}^{(1)} - w_{0,x}^{(2)}, \quad \hat{M}^{(1)} = \hat{M}^{(2)} \tag{5.3}$$

$$\hat{S}_{nr}^{(1)} - \hat{S}_{nr}^{(2)} = i(q_0^{(1)})^{-3} (w_{0,xxx}^{(1)} - w_{0,xxx}^{(2)}) \text{ when } x^{(k)} = x_*^{(k)}(0)$$

It follows from (5.2), that  $C_1^{(k)} = C_2^{(k)} = 0$ . Hence, on the freely supported edges, the principal terms in the series expansions of the edge-effect functions vanish.

Equations (5.3) are a system of linear inhomogeneous algebraic equations for determining the four constants  $C_3^{(k)} = C_4^{(k)}$ .

In the non-resonant case (3.11),  $w_0^{(2)} \equiv 0$  and the solution of system (5.3) has the form

$$C_3^{(1)} = C_4^{(2)} = \frac{\alpha_0^{(1)}}{4} D^{(1)} r_{43}, \quad C_3^{(2)} = C_4^{(1)} = \frac{\alpha_0^{(1)}}{4} D^{(1)} r_{34} \tag{5.4}$$

$$r_{43} = r_4 - r_3 / \sqrt{3}, \quad r_{34} = r_3 - r_4 / \sqrt{3}$$

From the condition for the boundary-value problem (5.1) to be solvable when  $k = 1$ , after some reduction and taking account of (2.9), (4.5) and (5.4), we obtain

$$\lambda_2 = \frac{1}{3l_0^{(1)}} \left( \frac{4\pi^2}{l_0^{(1)}} + \frac{169}{18} \text{tg}\beta \right) - \frac{7}{6} + \Lambda_2$$

$$\Lambda_2 = \frac{1}{3l_0^{(1)}} \left( \frac{\sqrt{2}(\sqrt{3}+1)\pi}{3^{1/4} l_0^{(1)}} - \frac{4}{\sin 2\beta} \right)$$

Consequently, in the non-resonant case

$$\lambda \approx \lambda_0^{(1)} + \varepsilon \lambda_1^{(1)} + \varepsilon^2 \lambda_2 \tag{5.5}$$

We note that  $\lambda_2 \rightarrow \infty$  when  $\beta \rightarrow 0$  and hence, for small  $\beta$ , the accuracy of the calculation is reduced. In the resonant case (3.12)

$$C_3^{(1)} = C_4^{(2)} = \frac{\alpha_0^{(1)}}{4} (D^{(1)} + D^{(2)}) r_{43}, \quad C_3^{(2)} = C_4^{(1)} = \frac{\alpha_0^{(1)}}{4} (D^{(1)} + D^{(2)}) r_{34}$$

and, from the conditions for the boundary-value problem (5.1) to be solvable, it follows that

$$\lambda_2^{(1)} = \lambda_2 + \Lambda_2 z, \quad \lambda_2^{(2)} = \lambda_2 + \Lambda_2 z^{-1}, \quad z = D^{(2)} / D^{(1)}$$

Let us assume that  $\Lambda_2 \sim 1$ . The magnitude of  $z$  is then determined from the condition

$$\lambda_0^{(1)} = \varepsilon \lambda_1^{(1)} + \varepsilon^2 \lambda_2^{(1)} = \lambda_0^{(2)} + \varepsilon \lambda_1^{(2)} + \varepsilon^2 \lambda_2^{(2)}$$

which reduces to a quadratic equation which has two real roots

$$z_{1,2} = p \pm \sqrt{p^2 + 1} > 0, \quad p = 2\pi\Delta / [3^{3/4} (l_0^{(1)})^2 \Lambda_2]$$

In the resonant case

$$\lambda = \begin{cases} \lambda_1^*, & \Lambda_2 < 0 \\ \lambda_1^*, & \Lambda_2 > 0 \end{cases} \tag{5.6}$$

$$\lambda_k^* = \lambda_0^{(1)} + \varepsilon \lambda_1^{(1)} + \varepsilon^2 (\lambda_2 + \Lambda_2 z_k)$$

and the form of the loss of stability with  $z = z_k$  corresponds to  $\lambda_k^*$ .

When  $\Delta = 0$ , in the case when  $\Lambda_2 < 0$ , the form of the loss of stability will be symmetric ( $D^{(1)} = D^{(2)}$ ). When  $\Delta \rightarrow \infty$ , formula (5.6) reduces to (5.5) and it can therefore also be used in the non-resonant case.

If  $\Lambda_2 = O(\epsilon)$  and, in particular,  $\Lambda_2 = 0$ , an approximate value of  $\lambda$  is found using formula (5.5) and the next approximations have to be constructed to determine  $z$ .

It follows from (2.3), (3.2), (3.9) and (4.5) that the form of the loss of stability can be approximately represented in the form

$$w = \begin{cases} w^{(1)}, & x_0^{(1)} \leq x^{(1)} \leq x_*^{(1)} \\ w^{(2)}, & x_*^{(2)} \leq x^{(2)} \leq x_0^{(2)} \end{cases}$$

where

$$w^{(k)} \approx \left[ 1 + \frac{4is_1^{(k)}}{3} \varphi - \left( \frac{103s_2^{(k)}}{36} - \frac{1}{32} \right) \varphi^2 \right] W_0^{(k)} \exp\left(\frac{Z^{(k)}(\varphi)}{\epsilon}\right)$$

$$Z^{(k)}(\varphi) = iq_0^{(k)} \varphi \left( 1 + \frac{11s_2^{(k)}}{18} \varphi^2 \right) - \frac{t_1^{(k)} \varphi^2}{2} \left[ 1 + \left( \frac{10s_2^{(k)}}{9} - \frac{1}{48} \right) \varphi^2 \right]$$

$$t_1^{(k)} = \frac{1}{4} \sqrt{\frac{\lambda_0^{(k)} t g \beta}{l_0^{(k)}}}, \quad s_1^{(k)} = \frac{t_1^{(k)}}{q_0^{(k)}}, \quad s_2^{(k)} = (s_1^{(k)})^2$$

The functions  $w^{(k)}$  are localized in the neighbourhood of the generatrix  $\varphi = 0$  and oscillate rapidly when  $\varphi$  varies. In the non-resonant case  $w^{(2)} \ll w^{(1)}$  and, consequently, the form of the loss of stability is localized in the neighbourhood of the generatrix  $\varphi = 0$  of the longer first shell.

The eigenvalue (5.7) is asymptotically twofold [3] since two real forms of oscillations  $\text{Re } w$  and  $\text{Im } w$  correspond to it. This means that, in a small neighbourhood of the approximate value of (5.6), there are two eigenvalues of boundary-value problem (1.1)–(1.5) and the difference between them is  $O(\epsilon^N)$  for any  $N$ .

Analogueous results can also be obtained for other homogeneous boundary conditions on the straight edges  $x^{(k)} = x_0^{(k)}$  of the joined shells.

6. The results of calculations of the upper critical pressure parameter  $\lambda$  for joined cylindrical shells with hinged supported edges are shown in Tables 1 and 2.

Table 1 illustrates the accuracy of the various asymptotic approximations as a function of the magnitude of  $R/h$  for shells with the following values of the parameters:  $l_0^{(1)} = 4$ ,  $l_0^{(2)} = 3$ ,  $\beta = 45^\circ$ ,  $\nu = 0.3$ . The zeroth approximation for  $\lambda$ , obtained using formula (3.10), is given in the first row. The results of calculations using formula (4.9), taking account of the first approximation, are given in the second row while the third row gives the results of calculations using formula (5.6) and taking account of the second approximation. The values of the quantity  $z^{-1} = D^{(1)}/D^{(2)}$  are shown in the fourth row.

Table 1

$R/h$	100	300	500
(0)	1.378	1.378	1.378
(1)	1.654	1.587	1.562
(2)	1.670	1.597	1.570
$z^{-1}$	106	171	215

Table 2

$l_0^{(2)}$	3	4	5
$\beta = 25^\circ$	1.554	1.541	1.221
$\beta = 35^\circ$	1.611	1.602	1.267
$\beta = 45^\circ$	1.670	1.662	1.314
$\beta = 55^\circ$	1.741	1.732	1.371



The dependence of  $\lambda$  on the length of the second shell  $l_0^{(2)}$  and the joining angle  $\beta$ , when  $R/h = 100$ ,  $l_0^{(1)} = 4$  and  $\nu = 0.3$ , is investigated in Table 2. As  $\beta$  increases, the critical pressure increases. The dependence of  $\lambda$  on  $l_0^{(2)}$  has a more complex form. As long as  $l_0^{(2)} < l_0^{(1)}$ , the critical loading parameter barely changes, since it is close to the critical loading parameter for the first shell with hinged supported edges. When  $l_0^{(2)} > l_0^{(1)}$ , the parameter  $\lambda$  decreases as  $l_0^{(2)}$  increases.

In the case of moderately thin shells ( $R/h < 200$ ), the stability calculation should be supplemented with a calculation of the stress-strain state since the maximum stress in the neighbourhood of the line where the shells are joined can reach the limiting value earlier than when loss of stability occurs. The stresses can be determined using the method of separating the stress-strain state into a zero-moment effect and a simple boundary effect [4].

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